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A note on possible flow instabilities in melting from the side

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Abstract — This paper presents a scale analysis of the parameters for which flow instabilities can be expected in coupled phase change - natural convection in cavities heated from the side. The results are in agreement with what was observed numerically for very low Prandtl number fluids, where multiple cells were found to appear early on as a result of the instability of the conduction regime. For very large Prandtl number flow instabilities, multiple cells, if any, would only occur at very large nominal Rayleigh number. © Elsevier, Paris.

flow instabilities / phase change / natural convection / moving boundaries / conjugate heat transfer

Résumé — Une note sur de possibles instabilités hydrodynamiques en couplage fusion-convection avec chauffage latéral. On présente une analyse d'ordres de grandeur des paramètres pour lesquels les écoulements de convection naturelle consécutifs à la fusion d'un matériau chauffé latéralement peuvent présenter des instabilités hydrodynamiques. Ces résultats sont confirmés par des simulations numériques dans des fluides de faible nombre de Prandtl, où des cellules transverses, correspondant à l'instabilité du régime de conduction, ont été observées pour des temps courts. Dans des fluides à fort nombre de Prandtl, de possibles instabilités ne pourraient être observées qu'à des nombres de Rayleigh nominaux très grands. © Elsevier, Paris.

instabilité hydrodynamiques / changement de phase / convection naturelle / frontière mobile / transferts couplés

Nomenclature

A	aspect ratio of the enclosure, $= H/L$	
C	scaling factor	
C_P	specific heat of the liquid phase	$J \cdot kg^{-1} \cdot K^{-1}$
Fo	Fourier number, $= \alpha t/H^2$	
g	acceleration of gravity	$m \cdot s^{-2}$
Gr_H	Grashof number, $g \beta \Delta T H^3/\nu^2$	
H	height of the enclosure	m
k	thermal conductivity of the liquid	$W \cdot m^{-1} \cdot s^{-1}$
L	width of the enclosure	m
L_F	latent heat	$J \cdot kg^{-1}$
Nu	average Nusselt number	
Pr	Prandtl number, $= \nu/\alpha$	
Ra_H	Rayleigh number, $= Pr\, Gr_H$	
Re	Reynolds number, $= (w\,\delta)/\nu$	
Ste	Stefan number, $= C_P(T_1 - T_F)/L_F$	
t	dimensional time	s
T	dimensional temperature	K
T_F	melting temperature	K
T_0	initial temperature	K
T_1	hot wall temperature	K
T_m	reference temperature $= (T_F + T_0)/2$	K
$w\,(u)$	vertical (horizontal) component of velocity	$m \cdot s^{-1}$
$x\,(z)$	dimensionless coordinates, $= x^*/H\ (z^*/H)$	

Greek symbols

α	thermal diffusivity	$m^2 \cdot s^{-1}$
β	coefficient of volumetric thermal expansion	K^{-1}
ΔT	temperature difference between walls, $T_1 - T_F$	K
δ	instantaneous width of liquid region	m
ν	kinematic viscosity	$m^2 \cdot s^{-1}$
τ	dimensionless time, $= Fo \times Ste$	
θ	dimensionless temperature, $\theta = (T - T_m)/\Delta T$	

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1. INTRODUCTION

Many industrial or natural processes involve solid/liquid phase change phenomena. One prototype configuration to study the influence of natural circulation on the phase change kinetics is the so-called differentially heated cavity where a piece of solid material is suddenly heated above its melting temperature along a vertical boundary. This problem has been addressed many times in the literature, experimentally, analytically and numerically. In particular many numerical algorithms have been derived to simulate this configuration, which fall into two categories, adaptive grids or fixed grids. Both classes of solution methodology have advantages and drawbacks in terms of accuracy and computational efficiency.

In order to compare these methods on an equal footing, we recently proposed a benchmark exercise to assess the performance and predictions of numerical algorithms designed to tackle such a coupled problem [3]. For the geometrical configuration defined in *figure 1*, four test cases were defined corresponding to two values of the Rayleigh number for two Prandtl numbers. The Prandtl numbers were chosen to match the value of gallium or tin ($Pr = 0.02$) and octadecane ($Pr = 50$). For each Prandtl number, the higher Rayleigh number was computed to correspond to actual experiments for which experimental measurements are available.

In the course of the computations, two of the participants to the benchmark predicted a somewhat slower Nusselt number decrease with time for the case labelled # 2 ($Ra_H = 2.5{\cdot}10^5$, $Pr = 0.02$; *figure 2*). This was attributed to the appearance of multiple cells early in the development of the solution. Four cells were found to establish at onset, which quickly reduced to 3 by the merging of the two uppermost, and then to 2 again after the merging of the two uppermost. It was also found that the two-cell regime became oscillatory, in the sense that a high frequency oscillation developed on top of the slow transient, corresponding to the melting time scale. This multicellular structure was found to quite appreciably enhance the overall heat transfer and to substantially affect the shape of the interface, appearing not just as a consequenceless curiosity. It was later brought to our attention that multiple cells were also found by Dantzig [5] in a finite element simulation of the experiment by Gau and Viskanta [7]. It should be noted however that Gau and Viskanta do not mention any such phenomenon (cells and/or high frequency unsteadiness) in their experiments. Later experiments in the same parameter range by Campbell and Koster [4] with a more accurate visualization of the interface also did not mention such effects.

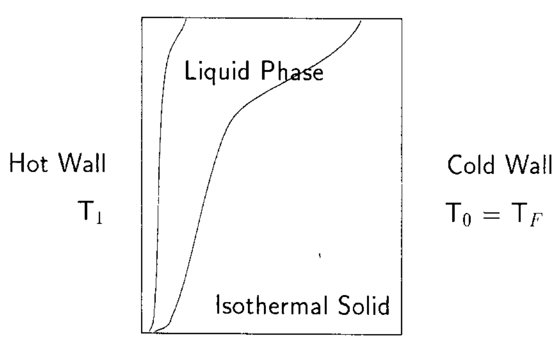

Figure 1. Schematic diagram of the problem.

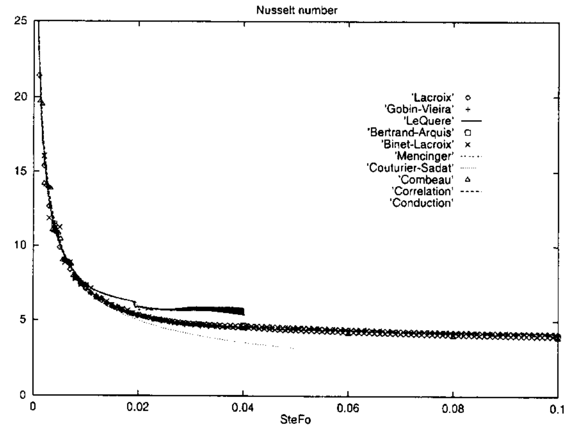

Figure 2. Time evolution of the average Nusselt number for case # 2. The different curves are labelled according to the names of the benchmark participants.

The purpose of this note is to establish criteria governing the possible onset of multicellular flow in the early times when melting a pure substance from a heated wall. The paper is organized as follows. We first briefly recall the problem definition. We then address the problem of multicellular instability, and from scaling laws and available stability analyses we establish general criteria governing the possible onset of multicellular flow. We finally address the question of the onset of oscillatory instability.

2. PROBLEM DEFINITION

The problem under consideration deals with the the melting of a pure substance controlled by natural convection in the melt. One considers a 2D cavity (height H and width L) initially filled with a solid material uniformly at the melting temperature ($T_0 = T_F$). At $t = 0$, the temperature of one of the vertical walls (the left wall in *figure 1*) is raised to a value $T_1 > T_F$, while the other vertical wall is maintained at the initial temperature. The horizontal walls are assumed to be adiabatic and no-slip. The fluid flow is supposed to be in the laminar regime, and the thermophysical properties of the material (thermal diffusity α, kinematic viscosity ν, thermal capacity C_P, latent heat L_F) to be constant.

After a pure conduction stage, thermal convection develops in the liquid phase, causing a non-uniform distribution of the heat flux at the interface and a non-uniform displacement of the melting front.

The problem is characterized by the following parameters:

- The Rayleigh number: $Ra_H = g\,\beta\,(T_1 - T_F)\,H^3/\alpha\,\nu$
- The Prandtl number: $Pr = \nu/\alpha$
- The Stefan number: $Ste = C_P(T_1 - T_F)/L_F$
- The global aspect ratio: $A = H/L$.

3. PRIMARY INSTABILITY

3.1. Scaling analysis

Early on the fluid motion develops in a tall cavity. It is known that this type of fluid circulation may be prone to hydrodynamic instabilities classically known as the instability of the conduction regime. The onset of such instabilities has been generally quantified as a function of the Grashof ($g\,\beta\,\Delta T\,L^3/\nu^2$) or Rayleigh ($Ra = Gr\,Pr$) numbers, based on the width L of the rectangular fluid enclosure. It is known that for the conduction solution the fluid velocity expression is

$$w = \frac{1}{6}\frac{g\,\beta\,\Delta T\,L^2}{\nu}\,x\,(x - 0.5)(x + 0.5) \qquad (1)$$

where x represents the horizontal length scaled with the width L of the cavity and ranges from -0.5 to 0.5. The corresponding maximum fluid velocity is approximately equal to $0.008\,g\,\beta\,\Delta T\,L^2/\nu$. The Grashof number can thus be viewed as a Reynolds number based on the width of the layer and on the characteristic scaling for the velocity. If the Reynolds number is based on the maximum velocity the relationship reads $Re \simeq Gr/125$. The fluid layer will then become linearly unstable when its characteristic Reynolds number reaches a critical value $Re_c \simeq Gr_c(Pr)/125$.

Let us assume that the Stefan number is very small. In this limit, the temperature field across the slot is in the conduction regime and the temperature gradient is $\Delta T/\delta$, where δ is the width of the liquid region. A conduction-phase change balance at the fluid-solid interface classically yields

$$\delta^2 = 2\,\alpha\,Ste\,t \qquad (2)$$

Going to the vertical-momentum equation, the dominant terms are the unsteady, viscous and buoyancy terms (see e.g. [1]), which are, upon using (2), of respective order

$$\frac{w}{t},\ \frac{Pr}{Ste}\,\frac{w}{t},\ g\,\beta\,\Delta T$$

In most situations of practical interest $Pr/Ste \gg 1$ (in the present cases, Pr/Ste is respectively 2 and 500) and consequently

$$w = C\,\frac{g\,\beta\,\Delta T\,Ste}{Pr}\,t \qquad (3)$$

The scaling factor C was determined from the computations for cases # 2 and # 4 (*figure 3*) and was found to be equal to 0.015. The instantaneous characteristic Reynolds number, based on the instantaneous velocity maximum and layer width, thus reads:

$$Re(t) \simeq 0.02\,\frac{Ra_H}{Pr}\left(\frac{t}{\dfrac{H^2}{\alpha\,Ste}}\right)^{3/2} \qquad (4)$$

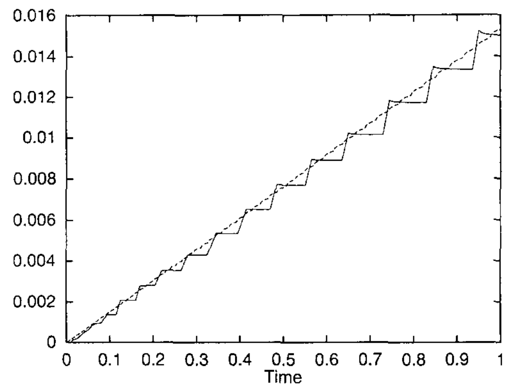

Figure 3. Early time evolution of maximum of the vertical velocity component at $z = 1/2$: (- - -) case # 2; (—) case # 4. The velocity unit is $\frac{\alpha}{H}\,Ra_H^{1/2}$ and the time unit is $\frac{H^2}{\alpha}\,Ra_H^{-1/2}\,St^{-1}$. The scaling factor in equation (3) is the average slope of the lines.

One can thus anticipate that instabilities will occur when time increases and the critical time at which the layer will become linearly unstable is

$$t_c \simeq 0.5\,\frac{H^2}{\alpha\,Ste}\left(\frac{Gr_c}{Gr_H}\right)^{2/3} \qquad (5)$$

By this time the cavity aspect ratio is

$$A_c \simeq \left(\frac{Gr_H}{Gr_c}\right)^{1/3} \qquad (6)$$

The stability of the conduction solution for each of the Prandtl numbers considered in the benchmark problem has been investigated by various authors [2, 8, 11]. The critical Grashof number for $Pr = 0.02$ is 8 000 and the most dangerous modes are steady, while for $Pr = 50$ the critical Grashof number is of the order of 1 000 and the instability is oscillatory. The corresponding critical times and aspect ratios are listed in the *table I*. The fact that the numerical values vary very little for each Prandtl number is purely accidental.

TABLE I
Critical dimensionless times and aspect ratios: top for $Pr = 0.02$, bottom for $Pr = 50$.

Ra_H	A_c	τ_c
$2.5{\cdot}10^4$	5	0.02
$2.5{\cdot}10^5$	11	0.004
Ra_H	A_c	τ_c
$1{\cdot}10^7$	5.5	0.016
$1{\cdot}10^8$	12	0.0035

The above analysis has obvious limitations. It first assumes that the base flow solution is quasi-steady and that the instability develops rapidly compared to the slow time scale evolution. Extending the linear stability analysis for an unsteady base flow solution is however beyond the scope of this note. The time given by (5) is thus an underestimation of the actual time at which the instability should become visible. Secondly, the instability will develop provided the flow is still in the conduction regime. One may anticipate that two main reasons can prevent this. The first one is non parallelism: if the aspect ratio is too small, the fluid cavity will already be highly distorted which may prevent the instability. The other one is due to the well known fact that stratification may develop, in particular for large Prandtl numbers, and make the flow switch from the conduction to the improperly named 'transition' or separated boundary layer regimes. The stability of the transition and boundary layer regimes may be quite different from those of the conduction regime, as described by Bergholz [2] who showed that the critical Grashof number is an increasing function of the stratification parameter.

3.2. Experimental verification

3.2.1. Low Prandtl number

In view of the above analysis, the numerical results were checked and it was indeed found that at time $\tau = 0.0042$, the solution corresponding to $Pr = 0.02$ and $Ra_H = 2.5{\cdot}10^5$ showed the formation of four cells, regularly spaced throughout the cavity height (*figure 4*). No cells were found either for $Ra_H = 2.5{\cdot}10^4$, or for $Pr = 50$ whatever the value of Ra_H. The reason why no cell was found for $Ra_H = 2.5{\cdot}10^4$ is very likely due to the fact that the instability would occur at a time of 0.02 when the cavity aspect ratio is 5. At this late time, non parallelism is already strong and is likely to prevent the instability to appear. Concluding that the instability can only develop when the cavity aspect ratio is still larger than 10 leads to a general criterion governing the onset of instabilities for small Prandtl number fluids

$$Gr_H \geqslant 5{\cdot}10^6 \sim 10^7$$

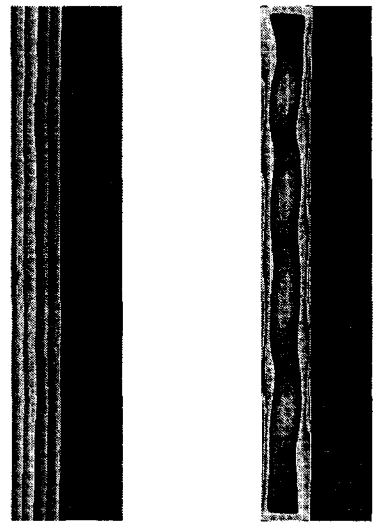

Figure 4. Temperature and vorticity fields at $\tau = 0.0048$.

3.2.2. Large Prandtl number

For $Pr = 50$ no instability was found whatever the value of the Rayleigh number, even the higher one. The explanation is that in large Prandtl number fluids, there develops a slight vertical stratification very early as the Grashof number is increased (due to the low thermal diffusivity), which makes the flow switch to the transition or boundary layer regime. At large Prandtl numbers, the situation is in fact very complex from the standpoint of possible instabilities. In infinite slots, as shown by Bergholz [2] amongst others, the conduction regime is prone to a multicellular stationary instability if $Pr \leq 12.7$ and to an oscillatory instability if $Pr \geq 12.7$. When $Pr \geq 12.7$ this mode of instability persists for small stratifications until, for large enough Prandtl number (50 seems to be the critical value), stationary modes become more dangerous (see [2], *figure 11*). Nobody has however observed the transition to this oscillatory regime in finite cavities of large aspect ratio although, as noted by Lee and Korpela [10], numerical computations have been performed such that their representative point plotted in a (stratification parameter, Grashof number) diagram is well inside the instability region. On the other hand numerical computations and experimental visualizations of very large Pr ($Pr \simeq 1\,000$) flows performed in moderately large aspect ratio cavities ($A \simeq 20$) have long shown multicellular stationary patterns [6, 12]. It can thus been inferred that the critical aspect ratio needed to see this multicellular stationary instability increases rapidly as the Prandtl number decreases. In particular, Bergholz estimated that it would require a cavity of aspect ratio on the order of 100 to see the appearance of the steady multicellular regime in water. The conclusion is that it is thus not clear which type of instability could be seen in the case of high Prandtl number fluids.

Let us just show that the instability of the conduction regime is very unlikely. The end of the conduction regime was quantified to be $Ra_H \leq 300\,A^4$ $(Pr > 1)$ [6, 9]. Inserting the aspect ratio given by (6) in this expression yields a critical value

$$Ra_H = \frac{1}{300^3}\,(Pr\,Gr_c)^4$$

Using the characteristic values for $Pr = 50$ gives a nominal Rayleigh number larger than 10^{11}. Since $Pr\,Gr_c$ is an increasing function of Pr over the range 20 – 1 000 [2], it is thus clear that high Prandtl number flows will be very unlikely to display the conduction regime instability in this material processing context, although possibly in some geophysical configurations. There is still one chance that it could however display the steady multicellular instability at large enough Prandtl number ($Pr \simeq 1\,000$) provided the Stefan number is small enough for the parallel flow approximation to remain valid long enough to let the instability develop.

4. SECONDARY OSCILLATORY INSTABILITY

Also clearly visible in *figure 2* is the fact that the mean Nusselt number, and hence the numerical solution becomes unsteady at a dimensionless time close to 0.03. This instability seems to originate in the shear between the two main rolls. In order to determine whether this unsteadiness is due to the coupling between natural convection and phase change, possibly induced by a numerical instability, or simply intrinsic to the natural convection flow regime alone, we have carried out a numerical integration of the Boussinesq equations in a differentially heated cavity of vertical aspect ratio equal to 3, of the same order as that characterizing the aspect ratio of the coupled problem at the onset of unsteadiness. Starting from rest, the time evolution of the temperature at a given monitoring point is displayed in *figure 5*. It indeed shows that the asymptotic solution is time periodic, very likely due to a Hopf bifurcation of the steady solution. In the convective time unit $((H^2/\alpha)\,Ra_H^{-1/2})$ used for the computation, the period of these oscillations is 18.1, which compares favorably with that of 10.8 characterizing the oscillations of the coupled problem. (Note that in a cavity of aspect ratio 4, the time asymptotic solution is also periodic with a period of 13). Snapshots of instantaneous temperature fields corresponding to the pure natural convection and to the coupled problem are presented in *figure 6*, showing the qualitative ressemblance and thereby establishing that the oscillatory instability is intrinsic to the natural convection alone.

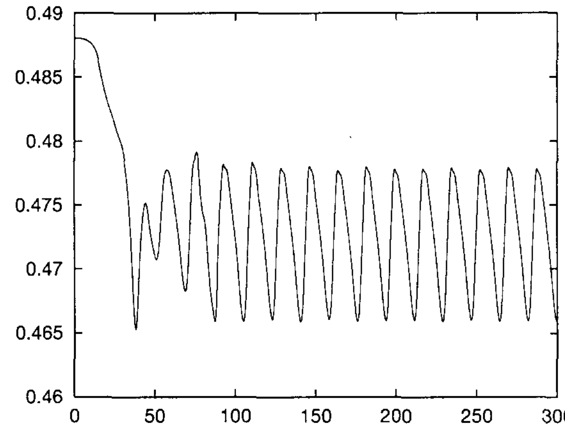

Figure 5. Time evolution of temperature θ at $(x/H, z/H) = (0.025,0.5)$ for $Ra_H = 2.5{\cdot}10^5$, $H/W = 3$. The reference time for this computation is $H^2/\alpha\,Ra_H^{1/2}$.

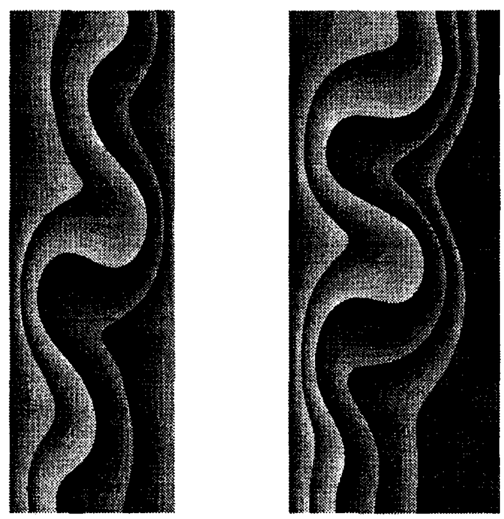

Figure 6. Instantaneous temperature field in the time periodic asymptotic regime for pure natural convection (left). Instantaneous temperature field of coupled phase change natural convection configuration at time $\tau_3 = 0.04$ (right). The rightmost contour line is the front position.

5. CONCLUSION

We have shown that the natural convection flow resulting from melting of pure low Prandtl number substances heated from the side is prone to the classical multicellular instability for sufficiently large nominal Rayleigh numbers. When it happens, this instability strongly influences the rate of heat transfer and hence the melting speed and also strongly influences the shape of the liquid/solid interface. The multi-roll regime that results from this initial steady instability becomes later also prone to an oscillatory instability, which is intrinsic to the natural convection flow alone. For large Prandtl number fluids, the instability of the conduction regime is very unlikely in the context of material processing, although one may expect that very large Prandtl number fluids could also display the multicellular instability for small enough Stefan numbers. These results and hypothesis require further detailed experimental investigations.

Acknowledgements

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